Ground-state entropies of the Potts antiferromagnet on diamond hierarchical lattices

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The ground-state degeneracies of the q-state Potts antiferromagnet on general diamond hierarchical lattices are computed, for $q \geqslant 3$, by means of two distinct methods. The first method, denominated the recursive approach, is based on exact recursion relations for the total number of ground states, leading to the exact ground-state entropy in the thermodynamic limit. The second method, called the factorization approach, consists in a simple approximation, where the total number of ground states is factorized as a product of the number of ground states at each hierarchy level. The factorization approach appears to be a poor approximation for small values of q, but its accuracy improves substantially as q increases, and it becomes exact in the limit $q \to \infty$. In spite of the fact that such a model presents no frustration, a residual entropy at zero temperature is found for all $q \geqslant 3$. Similarly to what happens on Bravais lattices, the residual entropy approaches its maximum allowed value, $\ln q$, as q increases.

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I. INTRODUCTION

The q-state Potts model [1] has attracted the attention of many workers since its original formulation [2]. Apart from representing a generalization of the most investigated system in statistical mechanics—the Ising model (the particular case $q=2$ of the Potts model)—it has been identified, for $q \neq 2$, with many other theoretical models, and has also been considered as the appropriate model for describing many physical phenomena [1]. Curiously, for the nearest-neighbor-interaction antiferromagnetic Potts model with $q>2$, one may easily see that the minimization of energy associated with any closed loop, at low temperatures, occurs with no conflict of interactions, in such a way that all interactions between nearest-neighbor spins remain satisfied. Therefore, the concept of frustration [3], which plays a central role in the ground-state (GS) degeneracy of some $q=2$ antiferromagnetic models, e.g., the antiferromagnetic Ising model on a triangular lattice, becomes irrelevant for $q \geqslant 3$. Even though there is no frustration, the antiferromagnetic Potts model may present, for sufficiently large values of q, a finite GS entropy per particle (usually denominated the residual entropy); this occurs due to the fact that one may have more than one choice of spin states that minimize the energy, at given sites of the lattice, leading to a multiplicity of ground states. That makes the antiferromagnetic Potts model an exception to the third law of thermodynamics.

Usually, for systems that present a large multiplicity of low-temperature states, the total number of GS's, N_{GS}, increases exponentially with the number of sites N,

$$N_{\mathrm{GS}} \sim \exp(hN), \tag{1.1}$$

where h is some positive finite number [for the Potts model, $0 \leqslant h \leqslant \ln q$, since the maximum number of states is $q^N = \exp(N \ln q)$]. In the thermodynamic limit, one gets that $h = s_0 / k_B$, where s_0 denotes the residual entropy.

The calculation of the GS entropy of the Potts antiferromagnet on Bravais lattices is a long-standing problem [4–7]. Although a lot of effort has been dedicated to this matter, most of the known results are due to approximations (see, e.g., Ref. [7] and references therein); only a few exact results are known, e.g., for the square lattice with the special case $q=3$ one has $s_0 = (3/2)\ln(4/3)$ [4], whereas for the triangular lattice, s_0 has been calculated for several values of q [5,6].

The study of magnetic models on fractal lattices, as well as serving in practice to model natural materials such as porous rocks, aerogels, sponges, etc., has provided useful results for the comprehension of the corresponding systems on Bravais lattices. In particular, the hierarchical lattices (HL's)—generated through recursive procedures—are much easier to handle [under the real-space renormalization group (RG)], in such a way that exact results may be obtained for short-range systems [8–10]. For pure systems defined on Bravais lattices, the RG procedure works as an approximation that may be implemented by means of a spin-decimation process which leads to RG equations. In the corresponding HL, such a procedure is exact for discrete classical spin variables, if within a few RG steps one gets nonproliferated RG equations connecting two successive hierarchy levels. Some particular HL's have been very successful in mimicking Bravais lattices [10], e.g., providing exact critical temperatures and exponents of magnetic models on the square lattice.

In the present work we calculate the GS degeneracy of the Potts antiferromagnet for the family of diamond HL's; we do that through the application of two different methods, previoulsy defined for Ising systems [11–13]. In the first method, denominated here the recursive approach (RA), one calculates the GS degeneracy recursively, through exact recursion relations based on the particular properties of the lattice. The main obstacle in the RA turns out to be in working out the recursion relations, which is not always feasible. In the sec-

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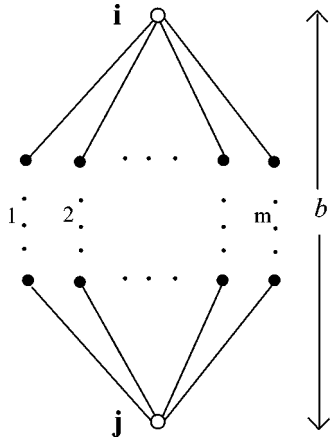

FIG. 1. The basic unit cell of the diamond HL considered; there are m parallel one-dimensional paths, each of them composed of b bonds in series (scaling factor b). The fractal dimension of such a lattice is $d=\ln(bm)/\ln b$. The spins at the terminal sites (empty circles) belong to previous hierarchy levels and are connected to other spins of the lattice; the spins at the internal sites (black circles) are to be decimated throughout the renormalization process. This basic unit cell corresponds to the HL at its hierarchy level $k=1$.

ond method, the factorization approach (FA), the total number of GS's at hierarchy level n is expressed as a product of properly defined partial number of GS's at hierarchy levels $n,n-1,\ldots,1$ [11]. In general, the FA is an approximation, leading to the exact result only for very simple systems [12], whereas in most cases it yields lower estimates [13] when compared with those obtained through the RA; however, it appears to be very useful, since it leads to a great simplification in the calculations and also due to the fact that it may provide accurate results in some cases [12]. The present paper is organized as follows. In the next section we discuss the RG transformation at zero temperature and the methods of calculation. In Sec. III we apply both methods to estimate the GS entropies of the q-state antiferromagnetic Potts model on general diamond HL's. Finally, in Sec. IV we present our conclusions.

II. THE MODEL AND FORMALISM

Let us consider the q-state antiferromagnetic Potts model, defined in terms of the Hamiltonian

$$\mathcal{H}=-J\sum_{\langle ij\rangle}\delta(\sigma_i,\sigma_j)\qquad(J<0,\sigma_i=1,2,\ldots,q),\quad(2.1)$$

where the sum $\Sigma_{\langle ij\rangle}$ is restricted to nearest-neighbor pairs of spins on a given diamond HL. At hierarchy level 0 one has $N^{(0)}=2$ sites connected by a single bond $N_b^{(0)}=1$, whereas at hierarchy level 1 one has a single cell, such as the one shown in Fig. 1; such a cell consists of m parallel paths connecting the external sites i and j, each path containing b bonds in series. The cell shown in Fig. 1 will be considered as the basic unit cell for the lattice at an arbitrary hierarchy level k, which is generated in such a way that at each step a single bond is replaced by a unit cell. The fractal dimension

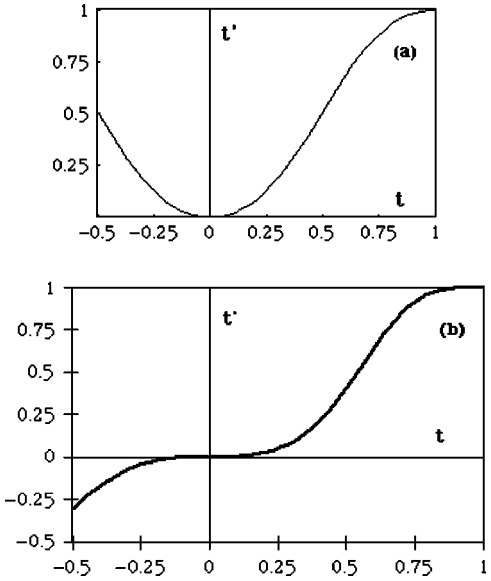

FIG. 2. Plots of the renormalized transmissivity t' versus the original transmissivity t for the q-state Potts antiferromagnet on diamond HL's: (a) Case $q=3$, $b=2$, and $m=2$; (b) case $q=3$, $b=3$, and $m=3$. The zero-temperature point ($t=-1/2$) is given by $a=\lim_{T\to 0}t'(T)=0.5$ in case (a), whereas $a\cong-0.306\,45$ in case (b). In both situations, a is different from the extreme values for the transmissivities, i.e., $a\neq-1/2,1$.

of such a HL is $d=[\ln(bm)/\ln b]$, whereas the number of bonds ($N_b^{(k)}$), cells ($N_c^{(k)}$), sites generated at level k ($\tilde{N}^{(k)}$), and total number of sites ($N^{(k)}$), at an arbitrary hierarchy level k, are given, respectively, by

$$N_b^{(k)}=(bm)^k,\quad N_c^{(k)}=(bm)^{k-1},\qquad(2.2)$$

$$\tilde{N}^{(k)}=(b-1)mN_c^{(k)},$$

$$N^{(k)}=2+(b-1)m\frac{(bm)^k-1}{bm-1}.\qquad(2.3)$$

One may easily obtain exact recursion relations $t'=f(t)$ for the thermal transmissivities [10]

$$t=\frac{1-\exp(-qJ/k_BT)}{1+(q-1)\exp(-qJ/k_BT)}\qquad(2.4)$$

of two successive hierarchy levels, t' [hierarchy level $(k-1)$] and t (hierarchy level k). It is important to mention that the thermal transmissivities, as defined above, are limited to the range $t\in[-1/(q-1),1]$. Such recursion relations lead to plots of t' versus t like the ones exhibited in Fig. 2.

Let us now consider, in such systems, the zero-temperature limit $T\to 0$; in general, one may write

$$t\approx a[1-c_1\exp(-c_2/T)],\qquad(2.5)$$

where the leading contribution a, which corresponds under the RG procedure to the zero-temperature point of our recur-

sion relation, i.e., for $t=-1$ one gets $a=\lim_{T\to 0} t'(T)$, may be calculated for the cell shown in Fig. 1,

$$a=\frac{x-y}{x+(q-1)y}, \tag{2.6}$$

where

$$x=(q-1)^m[(q-1)^{b-1}+(-1)^b]^m,$$

$$y=[(q-1)^b-(-1)^b]^m. \tag{2.7}$$

Under renormalization, the effective interactions among spins are given by

$$J'=-\lim_{T\to 0}\frac{k_B T}{q}\ln\left[\frac{1-t'(T)}{1+(q-1)t'(T)}\right], \tag{2.8}$$

which clearly implies that, whenever the argument of the logarithm is a finite constant, the interactions between spins are driven to zero, at the first renormalization step. The only exceptions to this rule may occur at the extreme values of the interval for the transmissivities, i.e., when $t'=1$ (argument of the logarithm is zero) and $t'=-1/(q-1)$ (argument of the logarithm diverges). A similar behavior has already been found in some fully frustrated Ising models, defined on special HL's [12]. Curiously, this effect occurs also for the antiferromagnetic Potts model—which does not present frustration—on diamond HL's.

Let us investigate in which cases the zero-temperature limit of the renormalized transmissivity, $a=\lim_{T\to 0} t'(T)$, assumes the extreme values mentioned above. From Eqs. (2.6) and (2.7) one gets that $a=1$ if $y=0$; the possible real solutions in this case are $q=0$ (b,m positive integers) and $q=2$ (b positive even integer, m positive integer). On the other hand, $a=-1/(q-1)$ occurs if $x=0$; in this case, the only possible real solution with $q>1$ appears to be the Ising case, $q=2$ (b positive odd integer, m positive integer). Therefore, for the q-state Potts antiferromagnet on general diamond HL's, one always has $a\neq -1/(q-1),1$, if $q\geqslant 3$ (typical examples are shown in Fig. 2). Other examples exhibiting a zero-temperature limit of the renormalized transmissivity, lying in between the two extremum values, were found for antiferromagnetic Potts models on different fractal lattices [14], as well as on a particular case of the diamond HL considered here [15]. Since the case $q=2$ has already been discussed for fully frustrated HL's [12], as well as for diamond HL's with random interactions (Ising spin glass) [11], we shall concentrate our analysis here in the cases $q\geqslant 3$.

Therefore, at zero temperature, the interactions between spins in our model are antiferromagnetic at the last hierarchy level n, and zero for all lower hierarchies, $n-1,n-2,\ldots,0$. Such curious zero-temperature behavior implies that the spins belonging to hierarchy levels $n-1,n-2,\ldots,0$, are completely uncorrelated (as in the high-temperature phase), contributing with the maximum number of states to the GS degeneracy. This is a crucial point for the calculation of the GS degeneracy, as will be seen later.

Let us now introduce the two methods for calculating the GS dengeneracy; we shall briefly review these methods, since they have been extensively discussed in previous work [12,13]. First of all, let us fix the terminal spins of each unit cell; for a Potts system, there are q^2 ways of doing this for a single cell on a diamond HL. For each fixed configuration of terminal spins in a unit cell, one may have a certain number of GS's associated with the internal spins of the cell. We shall denote by $\{g_\alpha(q,b,m)\}$ the possible set of GS degeneracies associated with the unit cell shown in Fig. 1, where the label α refers to configurations of terminal spins of the cell presenting different values of degeneracies. Due to the Potts symmetry, the unit cell shown in Fig. 1 may present only two types of degeneracies, i.e., $\alpha=1,2$; here, we will associate the degeneracies $g_1(q,b,m)$ and $g_2(q,b,m)$ with the cases of terminal spins in the same, and in different Potts states, respectively.

The RA [12] is based on the recursive properties of the HL; the central idea is to express GS degeneracies at a given hierarchy level in terms of those of the previous hierarchy. By fixing the spins of the hierarchy level 0, one has two possible degeneracies, at an arbitrary hierarchy level k, $G_1^{(k)}(q,b,m)$ and $G_2^{(k)}(q,b,m)$, associated with terminal spins in the same or in different Potts states, respectively. Such degeneracies follow the recursion relations

$$G_1^{(k)}(q,b,m)=\Psi_1(G_1^{(k-1)}(q,b,m),G_2^{(k-1)}(q,b,m)), \tag{2.9a}$$

$$G_2^{(k)}(q,b,m)=\Psi_2(G_1^{(k-1)}(q,b,m),G_2^{(k-1)}(q,b,m)). \tag{2.9b}$$

Since one may compute easily the set of degeneracies at hierarchy level 1, $\{G_\alpha^{(1)}\}\equiv\{g_\alpha\}$, the recursion relations above may be followed up to any desired hierarchy level. The total number of GS's of the HL at its nth hierarchy level is expressed as

$$N_{\mathrm{GS}}^{(n)}(q,b,m)=a_1 G_1^{(n)}(q,b,m)+a_2 G_2^{(n)}(q,b,m), \tag{2.10a}$$

$$G_1^{(n)}(q,b,m)=\Psi_1'(g_1(q,b,m),g_2(q,b,m)), \tag{2.10b}$$

$$G_2^{(n)}(q,b,m)=\Psi_2'(g_1(q,b,m),g_2(q,b,m)), \tag{2.10c}$$

where the coefficients a_α count how many different configurations of the spins at level 0 contribute to the same $G_\alpha^{(n)}$. For Potts models on diamond HL's, one has $a_1=q$ and $a_2=q(q-1)$.

If one succeeds in obtaining the recursion relations above exactly, the RA yields the exact number of GS's of the HL at its nth hierarchy level. Sometimes, working out such recursion relations turns out to be a difficult task; in such cases, one may use an approximation, the FA, which is a much simpler method, to be defined below.

One may partially count the number of GS's of the HL by fixing the terminal spins of each unit cell. We shall denote the number representing this partial counting by $\Gamma^{(n)}$. In a

HL all unit cells present terminal spins belonging to lowest-level hierarchies; under the RG procedure, each terminal spin will become an internal one at its respective hierarchy level. Therefore, one may write the number of GS's, calculated within the FA [11], as

$$N_{\mathrm{GSFA}}^{(n)}(q,b,m)=\langle\Gamma^{(n)}(q,b,m)\rangle\langle\Gamma^{(n-1)}(q,b,m)\rangle \times\langle\Gamma^{(n-2)}(q,b,m)\rangle\cdots\langle\Gamma^{(1)}(q,b,m)\rangle A, \tag{2.11}$$

where the factor $A=q^2$ corresponds to the number of states associated with hierarchy level 0, and $\langle\cdots\rangle$ stands for an average to be defined below. It is important to mention that, the total number of GS's may be factorized exactly in the above form only for very particular cases [12]; in most situations, Eq. (2.11) expresses a lower bound for the number of GS's, i.e., $N_{\mathrm{GSFA}}\leqslant N_{\mathrm{GS}}$ [13].

For an arbitrary hierarchy level k of a diamond HL, the partial counting may be written as

$$\Gamma^{(k)}(q,b,m)=[g_1(q,b,m)]^{N_{c,1}^{(k)}}[g_2(q,b,m)]^{N_{c,2}^{(k)}}, \tag{2.12}$$

where $N_{c,\alpha}^{(k)}$ denotes the number of unit cells with degeneracy g_α in the HL at its kth hierarchy level. In simple systems, $N_{c,\alpha}^{(k)}$ may be calculated exactly, whereas in more complicated problems one may replace $N_{c,\alpha}^{(k)}$ by the average value [11]

$$\phi_\alpha^{(k)}=N_c^{(k)}F_\alpha^{(k)}, \qquad \sum_\alpha \phi_\alpha^{(k)}=N_c^{(k)}, \tag{2.13}$$

where $F_\alpha^{(k)}$ represents the probability of finding a unit cell of type α at hierarchy level k. Such a procedure leads to the average estimate

$$\langle\Gamma^{(k)}(q,b,m)]\rangle=[g_1(q,b,m)]^{\phi_1^{(k)}}[g_2(q,b,m)]^{\phi_2^{(k)}} \tag{2.14}$$

used in Eq. (2.11). In the next section, we apply both methods to calculate the GS entropies of the q-state Potts antiferromagnet on general diamond HL's.

III. CALCULATION OF GS ENTROPIES

As discussed in the previous section, all cases $q\geqslant 3$ lead to zero effective interactions after the first RG iteration. Therefore, for the present model on a HL at the nth hierarchy level, the interactions will be considered as antiferromagnectic at $k=n$ and zero for all other hierarchy levels.

Let us now implement the RA, starting, as usual, with the case $n=1$, which consists in a single unit cell. Since the cell of Fig. 1 is composed by m independent parallel paths, all of them connecting the same external sites, the degeneracies associated with such a cell may be written as

$$G_1^{(1)}(q,b,m)\equiv g_1(q,b,m)=[L_1^{(1)}(q,b)]^m,$$
$$G_2^{(1)}(q,b,m)\equiv g_2(q,b,m)=[L_2^{(1)}(q,b)]^m, \tag{3.1}$$

where $L_\alpha^{(k)}(q,b)$ denotes the degeneracy of a single one-dimensional path connecting the external sites of the unit cell. The expressions for $L_\alpha^{(k)}(q,b)$ may be calculated easily, for small values of b; for $b=2$ one gets

$$L_1^{(1)}(q,2)=q-1, \qquad L_2^{(1)}(q,2)=q-2, \tag{3.2}$$

whereas for $b=3$

$$L_1^{(1)}(q,3)=(q-1)(q-2),$$
$$L_2^{(1)}(q,3)=q-1+(q-2)^2. \tag{3.3}$$

The degeneracy associated with a one-dimensional path, with a general value of b, may be calculated by decomposing such a path into smaller pieces. Obviously, there are several equivalent ways of carrying out such a procedure; here, we consider the decomposition of a one-dimensional path with b bonds into two pieces, one containing b' and the other b'' ($b',b''<b$ and $b'+b''=b$). One gets

$$L_1^{(1)}(q,b)=L_1^{(1)}(q,b')L_1^{(1)}(q,b'') +(q-1)L_2^{(1)}(q,b')L_2^{(1)}(q,b''), \tag{3.4a}$$

$$L_2^{(1)}(q,b)=L_1^{(1)}(q,b')L_2^{(1)}(q,b'') +L_2^{(1)}(q,b')L_1^{(1)}(q,b'') +(q-2)L_2^{(1)}(q,b')L_2^{(1)}(q,b''). \tag{3.4b}$$

Applying the decomposition procedure for the cases $b=4$ and $b=5$ one obtains, respectively,

$$L_1^{(1)}(q,4)=(q-1)^2+(q-1)(q-2)^2, \tag{3.5a}$$

$$L_2^{(1)}(q,4)=2(q-1)(q-2)+(q-2)^3, \tag{3.5b}$$

$$L_1^{(1)}(q,5)=(q-1)^2(q-2)+(q-1)(q-2) \times[(q-1)+(q-2)^2], \tag{3.6a}$$

$$L_2^{(1)}(q,5)=(q-1)[(q-1)+(q-2)^2] +(q-1)(q-2)^2+(q-2)^2[(q-1) +(q-2)^2]. \tag{3.6b}$$

Using the results above in Eqs. (3.1) and (2.10a), one gets the total number of GS's at hierarchy level $n=1$.

Let us now consider the hierarchy level $n=2$, at which each bond of the cell in Fig. 1 will become a unit cell. One has

$$G_1^{(2)}(q,b,m)=[L_1^{(2)}(q,b,m)]^m,$$
$$G_2^{(2)}(q,b,m)=[L_2^{(2)}(q,b,m)]^m, \tag{3.7}$$

where the degeneracies $L_\alpha^{(2)}(q,b,m)$ now present a dependence on the number of parallel paths m of the previous

hierarchy level. Such degeneracies may be easily calculated for small values of b, e.g., for $b=2$,

$$L_1^{(2)}(q,2,m)=[g_1(q,2,m)]^2+(q-1)[g_2(q,2,m)]^2, \tag{3.8a}$$

$$L_2^{(2)}(q,2,m)=2g_1(q,2,m)g_2(q,2,m)+(q-2)[g_2(q,2,m)]^2, \tag{3.8b}$$

and for $b=3$,

$$L_1^{(2)}(q,3,m)=[g_1(q,3,m)]^3+3(q-1)g_1(q,3,m)\times[g_2(q,3,m)]^2+(q-1)(q-2)[g_2(q,3,m)]^3, \tag{3.9a}$$

$$L_2^{(2)}(q,3,m)=3[g_1(q,3,m)]^2g_2(q,3,m)+3(q-2)g_1(q,3,m)[g_2(q,3,m)]^2+[(q-1)+(q-2)^2][g_2(q,3,m)]^3. \tag{3.9b}$$

For higher values of b, one may apply a similar decomposition recipe as the one employed for hierarchy level $n=1$ [see Eqs. (3.4)], by replacing each of the b bonds by a unit cell. It should be pointed out that the expressions for $b=3$ in Eqs. (3.9) may also be calculated using such a procedure, with a decomposition into two smaller pieces, $b'=1$ and $b''=2$. The case $b=4$ may be calculated through several different decompositions, e.g., the choices $b'=1$ and $b''=3$, or $b'=b''=2$, are equivalent, and lead to

$$L_1^{(2)}(q,4,m)=[g_1(q,4,m)]^4+4(q-1)(q-2)g_1(q,4,m)[g_2(q,4,m)]^3+6(q-1)[g_1(q,4,m)]^2[g_2(q,4,m)]^2+[(q-1)(q-2)^2+(q-1)^2][g_2(q,4,m)]^4, \tag{3.10a}$$

$$L_2^{(2)}(q,4,m)=4[g_1(q,4,m)]^3g_2(q,4,m)+4[(q-2)^2+(q-1)]g_1(q,4,m)[g_2(q,4,m)]^3+6(q-2)[g_1(q,4,m)]^2[g_2(q,4,m)]^2+[2(q-1)(q-2)+(q-2)^3][g_2(q,4,m)]^4. \tag{3.10b}$$

One can now generalize the RA for a diamond HL on its nth hierarchy level; the total number of GS's is given by

$$N_{\mathrm{GS}}^{(n)}(q,b,m)=a_1G_1^{(n)}(q,b,m)+a_2G_2^{(n)}(q,b,m), \tag{3.11}$$

where

$$G_1^{(n)}(q,b,m)=[L_1^{(n)}(q,b,m)]^m,$$

$$G_2^{(n)}(q,b,m)=[L_2^{(n)}(q,b,m)]^m. \tag{3.12}$$

Using the fact that $L_\alpha^{(n)}(q,b,m)$ are related to $G_\alpha^{(n-1)}(q,b,m)$ and $G_\alpha^{(n-1)}(q,b,m)=[L_\alpha^{(n-1)}(q,b,m)]^m$, one may obtain a recursion relation for the degeneracies $L_\alpha^{(k)}(q,b,m)$. For $b=2$ and $b=3$ one has generalizations of Eqs. (3.8) and (3.9), which are given, respectively, by

$$L_1^{(n)}(q,2,m)=[L_1^{(n-1)}(q,2,m)]^{2m}+(q-1)\times[L_2^{(n-1)}(q,2,m)]^{2m}, \tag{3.13a}$$

$$L_2^{(n)}(q,2,m)=2[L_1^{(n-1)}(q,2,m)L_2^{(n-1)}(q,2,m)]^m+(q-2)[L_2^{(n-1)}(q,2,m)]^{2m}, \tag{3.13b}$$

and

$$L_1^{(n)}(q,3,m)=[L_1^{(n-1)}(q,3,m)]^{3m}+3(q-1)[L_1^{(n-1)}(q,3,m)]^m[L_2^{(n-1)}(q,3,m)]^{2m}+(q-1)(q-2)[L_2^{(n-1)}(q,3,m)]^{3m}, \tag{3.14a}$$

$$L_2^{(n)}(q,3,m)=3[L_1^{(n-1)}(q,3,m)]^{2m}[L_2^{(n-1)}(q,3,m)]^m+3(q-2)[L_1^{(n-1)}(q,3,m)]^m[L_2^{(n-1)}(q,3,m)]^{2m}+[(q-1)+(q-2)^2]\times[L_2^{(n-1)}(q,3,m)]^{3m}. \tag{3.14b}$$

For higher values of b, one may use a general form for the decomposition procedure of Eqs. (3.4),

$$L_1^{(n)}(q,b,m)=L_1^{(n)}(q,b',m)L_1^{(n)}(q,b'',m)+(q-1)L_2^{(n)}(q,b',m)L_2^{(n)}(q,b'',m), \tag{3.15a}$$

$$L_2^{(n)}(q,b,m)=L_1^{(n)}(q,b',m)L_2^{(n)}(q,b'',m)+L_2^{(n)}(q,b',m)L_1^{(n)}(q,b'',m)+(q-2)L_2^{(n)}(q,b',m)L_2^{(n)}(q,b'',m). \tag{3.15b}$$

Iterating the equations above, one may obtain the GS entropy per spin in the thermodynmic limit,

$$s_0(q,b,m)=\lim_{n\to\infty}\frac{1}{N^{(n)}}\ln N_{\rm GS}^{(n)}(q,b,m). \tag{3.16}$$

Let us now turn to the FA; one gets for hierarchy level n [see Eqs. (2.12) and (3.1)]

$$\Gamma^{(n)}(q,b,m)=[L_1^{(1)}(q,b)]^{mN_{c,1}^{(n)}}[L_2^{(1)}(q,b)]^{mN_{c,2}^{(n)}}, \tag{3.17}$$

whereas for all previous hierarchies, a zero effective interaction leads to

$$\Gamma^{(k)}(q,b,m)=q^{\tilde{N}^{(k)}} \quad (k=1,2,\ldots,n-1), \tag{3.18a}$$

with

$$\sum_{k=1}^{n-1}\tilde{N}^{(k)}=(b-1)m\frac{(bm)^{n-1}-1}{bm-1}. \tag{3.18b}$$

The probabilities of finding cells of type 1 and 2 [see Eq. (2.13)], are given, respectively, by $F_1^{(n)}=1/q$ and $F_2^{(n)}=(q-1)/q$, and so we replace $N_{c,1}^{(n)}$ and $N_{c,2}^{(n)}$ by $\phi_1^{(n)}=(1/q)N_c^{(n)}$ and $\phi_2^{(n)}=[(q-1)/q]N_c^{(n)}$. Using these results in Eq. (2.11), one gets that

$$\ln[N_{\rm GSFA}^{(n)}(q,b,m)]=\frac{m}{q}(bm)^{n-1}\{\ln[L_1^{(1)}(q,b)]+(q-1)\ln[L_2^{(1)}(q,b)]\}+\left[2+(b-1)m\frac{(bm)^{n-1}-1}{bm-1}\right]\ln q, \tag{3.19}$$

leading to the GS entropy per spin in the thermodynamic limit,

TABLE I. Residual entropies of the q-state antiferromagnetic Potts model on diamond HL's with scaling factor $b=2$, for typical values of q. In each case, the upper value corresponds to the RA estimate $s_0(q,b,m)$, whereas the value below is the lower bound calculated by means of the FA, $s_0^{(\rm FA)}(q,b,m)$. The values of m ($m=b^{d-1}$) chosen are associated with HL's with fractal dimensions $d=2$, 3, 4, and ∞. The RA estimates were obtained by looking at the convergence of s_0 through the iteration of Eqs. (3.11)–(3.13); the convergence becomes faster as m increases, in such a way that the iteration process was carried up to $n=12$ ($m=2$), $n=10$ ($m=4$), $n=7$ ($m=8$), and $n=3$ (m large, e.g., $m=300$). The FA estimates were obtained through Eq. (3.20) (finite values of m) and Eq. (3.26) (limit $m\to\infty$); above, we present the FA results up to six decimal digits. The maximum allowed value for each residual entropy is $\ln q$, i.e., 1.098 612 . . . ($q=3$), 1.386 294 . . . ($q=4$), and 2.302 585 . . . ($q=10$).

$b=2$	$q=3$	$q=4$	$q=10$
$m=2$	0.549306(1)	0.969322(1)	2.145058(0)
	0.447939 . . .	0.942458 . . .	2.144061 . . .
$m=4$	0.607354(0)	0.973435(0)	2.120133(1)
	0.339494 . . .	0.868486 . . .	2.117640 . . .
$m=8$	0.649827(0)	1.030216(0)	2.112067(1)
	0.285271 . . .	0.831499 . . .	2.104430 . . .
$m\to\infty$	0.6925(1)	1.098(1)	2.195(1)
	0.231049 . . .	0.794513 . . .	2.091219 . . .

$$s_0^{(\rm FA)}(q,b,m)=\frac{bm-1}{b(b-1)qm}\{\ln[L_1^{(1)}(q,b)]+(q-1)\ln[L_2^{(1)}(q,b)]\}+\frac{1}{bm}\ln q. \tag{3.20}$$

In Tables I and II we present the residual entropies calculated through both methods, for typical values of q, and scaling factors $b=2$ (Table I) and $b=3$ (Table II). A few points are worth stressing, as we discuss below.

(i) The residual entropies increase with q, as expected, and one may observe that the ratios $s_0(q,b,m)/\ln q$ and $s_0^{(\rm FA)}(q,b,m)/\ln q$ increase with q, indicating that the residual entropies approach their maximum allowed values $\ln q$, similarly to what happens on Bravais lattices [7].

(ii) The RA yields the exact estimates for the diamond HL's discussed here. However, the iteration process may lead to numerical difficulties at large hierarchies; in spite of that, we succeeded in computing the residual entropies with a convergence up to five (in some cases, up to six) decimal digits.

(iii) As discussed elsewhere [13], the FA yields lower bounds for the residual entropies. It appears to be a poor approximation for small values of q, but it improves its accuracy for increasing values of q. Indeed, in the limit $q\to\infty$ the FA becomes exact, as we show below. Considering only the dominant terms in the limit $q\to\infty$, one gets

TABLE II. Residual entropies of the q-state antiferromagnetic Potts model on diamond HL's with scaling factor $b=3$, for typical values of q. In each case, the upper value corresponds to the RA estimate $s_0(q,b,m)$, whereas the value below is the lower bound calculated by means of the FA, $s_0^{(\mathrm{FA})}(q,b,m)$. The values of m ($m=b^{d-1}$) chosen are associated with HL's with fractal dimensions $d=2$, 3, 4, and ∞. The RA estimates were obtained by looking at the convergence of s_0 through the iteration of Eqs. (3.11), (3.12), and (3.14); the convergence becomes faster as m increases, in such a way that the iteration process was carried up to $n=8$ ($m=3$), $n=6$ ($m=9$), $n=4$ ($m=27$), and $n=3$ (m large, e.g., $m=100$). The FA estimates were obtained through Eq. (3.20) (finite values of m) and Eq. (3.26) (limit $m\to\infty$); above, we present the FA results up to six decimal digits. The maximum allowed value for each residual entropy is $\ln q$, i.e., 1.098 612 . . . ($q=3$), 1.386 294 . . . ($q=4$), and 2.302 585 . . . ($q=10$).

$b=3$	$q=3$	$q=4$	$q=10$
$m=3$	0.570818(0)	1.004491(1)	2.162111(0)
	0.550271 . . .	1.001753 . . .	2.162100 . . .
$m=9$	0.549804(0)	0.977181(0)	2.150429(0)
	0.504576 . . .	0.969708 . . .	2.150393 . . .
$m=27$	0.54930(1)	0.973037(1)	2.146594(1)
	0.489344 . . .	0.959027 . . .	2.146491 . . .
$m\to\infty$	0.54930(1)	0.97295(1)	2.1454(1)
	0.481728 . . .	0.953686 . . .	2.144540 . . .

$$L_1^{(1)}(q,b), L_2^{(1)}(q,b) \sim q^{(b-1)}. \tag{3.21}$$

Substituting into Eq. (3.20),

$$s_0^{(\mathrm{FA})}(q,b,m) \sim \frac{bm-1}{b(b-1)qm} q(b-1)\ln q + \frac{1}{bm}\ln q, \tag{3.22}$$

which leads to $s_0^{(\mathrm{FA})}(q,b,m)\sim \ln q$. A similar result may be obtained also through the RA; using Eq. (3.21) and iterating Eqs. (3.13)–(3.15), one concludes that

$$L_1^{(n)}(q,b,m), L_2^{(n)}(q,b,m) \sim q^{(b-1)}[L_2^{(n-1)}(q,b,m)]^{bm}, \tag{3.23}$$

and therefore

$$N_{\mathrm{GS}}^{(n)}(q,b,m) \sim a_2 G_2^{(n)}(q,b,m) \sim q^2[L_2^{(n)}(q,b,m)]^m. \tag{3.24}$$

Iterating Eq. (3.23), one gets

$$L_2^{(n)}(q,b,m) \sim q^{(b-1)[(bm)^n-1]/(bm-1)}, \tag{3.25}$$

which leads to $s_0(q,b,m)\sim \ln q$.

(iv) For fixed values of q and b, the residual entropies approach well-defined limits, as m increases. This may be easily seen within the FA, for which Eq. (3.20) leads to

$$\lim_{m\to\infty} s_0^{(\mathrm{FA})}(q,b,m) = \frac{1}{(b-1)q}\{\ln[L_1^{(1)}(q,b)] + (q-1)\ln[L_2^{(1)}(q,b)]\}. \tag{3.26}$$

Within the FA, the convergence toward the limit $m\to\infty$ occurs in such a way that the residual entropy always decreases for increasing values of m. In contrast to that, the way that such a convergence occurs within the RA may vary with both q and b. Whenever this convergence in the RA occurs by increasing the residual entropy, a large discrepancy of the FA estimate is obtained (see, e.g., the case $q=3$, $b=2$, of Table I); however, one may get rather accurate estimates through the FA, when the convergence toward the limit $m\to\infty$ of the RA occurs by decreasing the value of the residual entropy (see, e.g., the case $q=10$, $b=3$ of Table II).

IV. CONCLUSION

We have computed the ground-state entropies of an antiferromagnetic q-state Potts model on general diamond hierarchical lattices. Essentially, two methods were used for this calculation: the recursive approach, based on exact recursion relations for the number of ground states, and the factorization approach, which consists in factorizing the total number of ground states as a product of the number of ground states in eeach hierarchy level. Whereas the former is an exact procedure, the latter provides a lower bound for the ground-state entropy, with the advantage of a great simplification in the calculations. Apart from the fact that such a model presents no frustration, we have shown that for all $q\geqslant 3$ there is a residual entropy at zero temperature; in particular, the residual entropy approaches the maximum allowed value, $\ln q$ as q increases. Although the factorization approach appears to be a poor approximation for low values of q, its accuracy is substantially improved as q increases, in such a way that it becomes exact in the limit $q\to\infty$. The application of the present methods for the investigation of other models is encouraging and may produce insights into statistical systems defined on fractal lattices.

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